ON BEHAVIORS OF CELLULAR AUTOMATA WITH RULE 58 UNDER THE BOUNDARY CONDITION 0-1

Tatsuro SATO

Dept. of General Education

In this paper, we treat the behaviors of 1-dimensional finite cellular automata with a triplet local transition rule 58 having the fixed boundary condition 0-1. The behaviors of *CA-58*(n) were observed by computer experiments, and some formulae on the limit cycle and the transient length were found out. The purpose of this paper is to give theoretical proofs to above formulae.

***Key Words* :** cellular automata, limit cycle, transient length, boundary condition

1. INTRODUCTION

Cellular automata were initially introduced by J. von Neumann as theoretical models to demonstrate a system capable of self-reproducing organisms and universal computation in 1950s[1]. From 1980s downward, cellular automata have been realized again as a theoretical model of complex system by S.Wolfram and the other physicists. And many researchers have investigated and applied them. Cellular automata have very simple structure. While cellular automata obey simple transition rules, their behaviors are very complicated. The complication is caused by interaction between cells and is similar to behaviors of complex systems as fractal and chaotic phenomena. The importance of cellular automata seems to increase in the fields of mathematics, physics, biology, computer science, economics and so on.

Various cellular automata have been analyzed concerning one dimensional and two dimensional cellular automata, cellular automata with a cyclic cell array or a linear cell array and so on. Wolfram classified 1-dimensional cellular automata into four complex classes according to patterns generated by the synchronous dynamics, a homogeneous state, a set of separated simple stable or periodic structures, a chaotic pattern and complex localized structures which are sometimes long-lived[2),3)].

Investigating behaviors of cellular automata, we think that the transient length and the period length of limit cycle are important concepts. Cellular automata having limit cycles of period length 1 or 2 were investigated by Inokuchi *et al.*[4)]. So first, we aimed at cellular automata having limit cycles of period length 3[5),6),7),8),9),10),11)]. By computer experiments, we observed that there were seven cellular automata having limit cycles of period length 3 except cellular automata with threshold rules, 24, 27, 46, 57, 58, 130 and 152. And we investigated their transient length and period length of limit cycle. Secondly, we investigated cellular automata with threshold rules. There are 38 threshold rules except for symmetric, reverse and symmetric reverse rules, 0, 1, 2, 3, 4, 5, 7, 8, 10, 11, 12, 13, 14, 15, 19, 23, 32, 34, 35, 42, 43, 50, 51, 76, 77, 128, 136, 138, 140, 142, 160, 162, 168, 170, 178, 200, 204 and 232. And we reproved that their period lengths of limit cycles are 4 or less and we proved that their transient lengths are $3\times$ (*cell-size*)-4 or less[12),13)].

2. PRELIMINARIES

In this section, we define 1-dimensional finite cellular automata and introduce notations used in this paper.

Cellular automata treated in this paper have linearly ordered and finite number cells bearing with states 0 or 1. The next state of any cell depends on the states of the left cell, the cell itself and the right cell. In this section, we will formally define cellular automata *CA*-R(n) with rule number R of triplet local transition rule f and n cell array.

Let Q be a state set {0, 1} and n a positive integer. The complement of a state $a \in Q$ will be denoted by a^-, that is a^-=1-a. The *n-th* cartesian product of Q is denoted Q^n in other words, Q^n is the set of all *n-tuples* consisting of 0 and 1. For example,

Q^3={000, 001, 010, 011, 100, 101,110, 111}.

DEFINITION 2.1

The triplet local transition rule is a function $f : Q^3 \to Q$*, and the rule number* R *of* f *is defined by*

$$R = \sum_{a,b,c \in Q} 2^{4a+2b+c} f(abc).$$

Usually the triplet local transition rule of rule number R is denoted by *rule R* for short.

DEFINITION 2.2

Let $I=\{1, 2, \cdots, n\}$ *be a set of cells. We call n-dimensional vector space* Q^n *a configuration space. And a configuration is a vector* $x=(x_1, x_2, \cdots, x_n) \in Q^n$.

The positive integer n is called cell-size. Usually a vector $(x_1, x_2, \cdots, x_n)$ denotes $x_1 x_2 \cdots x_n$ for short.

DEFINITION 2.3

The global transition function δ *is defined as follows*;

$\delta(x_1 x_2 \cdots x_n)=(f(\alpha\, x_1\, x_2) f(x_1\, x_2\, x_3) \cdots f(x_{n-1}\, x_n\, \beta))$

where f *is rule* R *and* $\alpha, \beta \in Q$.

We call a pair (α, β) a boundary. We say that the boundary condition is cyclic if and only if $\alpha = x_n$ and $\beta = x_1$, and the boundary condition is fixed if and only if α and β are fixed. We say that the boundary condition is *a-b* if $\alpha=a$ and $\beta=b$.

$CA-R_{\alpha-\beta}(n)$ denotes an one-dimensional cellular automata such that n cells exist and its transition rule R and its boundary condition is α - β.

DEFINITION 2.4

Let x *be a configuration of* $CA-R_{\alpha-\beta}(n)$*. The configuration* x *is on a limit cycle of period length* T *if there exists a positive integer* s *such that* $\delta^s(x)=x$*, and* $T=min\{s \geq 1 | \delta^s(x)=x\}$*. And the configurations* $x(1), x(2), \cdots, x(T-1)$ *form a limit cycle of period length* T *if* $x(i+1)=\delta(x(i))$ *and* $x(T)=x(1)$*, and* $x(i)$ *is on a limit cycle of period length* T *where* $1 \leq i \leq T-1$.

A limit cycle of period length T is denoted by a T-cycle, in particular, a limit cycle of period length 1 is denoted by a fixed point. And a number of a limit cycle of period length T is denoted by $\gamma_T(n)$.

DEFINITION 2.5

$h(x)$ *is defined as follows* ;

$h(x)=min\{s \geq 0 | \delta^s(x)$ *is on a limit cycle*$\}$.

Then the transient length $H(n)$ of $CA-R_{\alpha-\beta}(n)$ is defined as $H(n)=max\{h(x)|x \in Q^n\}$.

DEFINITION 2.6

The symmetric transition rule $f^{\#} : Q^3 \to Q$ *of a local transition rule* f *is defined as follows*;

$f^{\#}(abc)=f(cba)$

for all triples $abc \in Q^3$.

DEFINITION 2.7

The reverse transition rule $f^{\circ} : Q^3 \to Q$ *of a local transition rule* f *is defined as follows* ;

$f^{\circ}(abc)=[f(\bar{a}\bar{b}\bar{c})]^{-}=1-f(1-a, 1-b, 1-c)$

for all triples $abc \in Q^3$.

Let rule R° and rule $R^{\#}$ be the reverse and symmetric rule of rule R respectively. It is trivial that $f^{\#\#}=f$ and $R^{\#}=R+56(r_3-r_6)+14(r_1-r_4)$ and $f^{\circ\circ}=f$, $f^{\#\circ}=f^{\circ\#}$ and $R^{\circ}=255-(128 r_0+64 r_1+32 r_2+16 r_3+8 r_4+4 r_5+2 r_6+ r_7)$ where $r_{4a+2b+c}=f(abc)$ for all triples $abc \in Q^3$. Then, we can identify $CA-R^{\circ}{}_{\bar{\alpha}-\bar{\beta}}(n)$ and $CA-R^{\#}{}_{\beta-\alpha}(n)$ with $CA-R_{\alpha-\beta}(n)$. So, $CA-R_{\alpha-\beta}(n)$, $CA-R^{\circ}{}_{\bar{\alpha}-\bar{\beta}}(n)$, $CA-R^{\#}{}_{\beta-\alpha}(n)$ and $CA-R^{\circ\#}{}_{\bar{\beta}-\bar{\alpha}}(n)$ can be identified with each other.

Considering the reverse rule, the symmetric rule and the symmetric reverse rule, 256 triplet transition rules can be classified into nonequivalent 88 groups.

In addition to above definitions, the notations in this paper are as follows;

- $a^k=\underbrace{aa\cdots a}_{k\text{-}times}$ (a=0 or 1).

Let A be a subsequence.

- $A^k=\underbrace{AA\cdots A}_{k\text{-}times}$.
- $[A)^*_l$: sequence composed of l bits taken from the left edge when some A's are arranged.
- $(A]^*_l$: sequence composed of l bits taken from the right edge when some A's are arranged.

- ＊ : an arbitrary bit.
- The state of i-th cell of a configuration x is denoted by x_i.
- x_0 and x_{n+1} mean the left and right boundary of x respectively. That is, in $CA-R_{\alpha-\beta}(n)$ $x_0=\alpha$ and $x_{n+1}=\beta$.

3. BEHAVIORS OF *CA* - $58_{0\text{-}1}(n)$

A finite cellular automaton CA-$58_{0\text{-}1}(n)$ treated in this section has the following triplet local transition rule f by the definition of the rule number:

$$\begin{vmatrix} 111 & 110 & 101 & 100 & 011 & 010 & 001 & 000 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \end{vmatrix}$$

because $58=2^5+2^4+2^3+2^0$.

LEMMA 3.1
For any configuration x of CA-$58_{0\text{-}1}(n)$, $\delta(x)$ *does not contain the subsequence* 1111.

Proof.
We set $\delta(x)=y$ and $y_iy_{i+1}y_{i+2}y_{i+3}=1111$ $(1\leq i\leq n-3)$. Then, we have $x_{i-1}x_ix_{i+1}=001$, 011, 100 or 101 for $y_i=1$.

(i) In the case $x_{i-1}x_ix_{i+1}=001$ or 101
For $y_{i+1}=1$, we have $x_{i+2}=1$. But this contradicts $y_{i+2}=1$ as $f(11*)=0$.

(ii) In the case $x_{i-1}x_ix_{i+1}=011$
This case contradicts $y_{i+1}=1$ as $f(11*)=0$.

(iii) In the case $x_{i-1}x_ix_{i+1}=100$
For $y_{i+1}y_{i+2}=11$, we have $x_{i+2}x_{i+3}=11$. But this contradicts $y_{i+3}=1$ as $f(11*)=0$. □

COROLLARY 3.2
For any configuration x of CA-$58_{0\text{-}1}(n)$, *we set* $\delta(x)=y$. *Then, we have* $y_1y_2y_3\neq 111$.

LEMMA 3.3
For any configuration x of CA-$58_{0\text{-}1}(n)$, *we set* $\delta^k(x)=y$ *where* $k\geq 2$. *Then, we have* $y_{n-2}y_{n-1}y_n\neq 111$.

Proof.
We set $y_{n-2}y_{n-1}y_n=111$ and $\delta^{k-1}(x)=z$. Then, we have $z_{n-3}z_{n-2}z_{n-1}z_n=1111$. But this contradicts lemma 3.1 as $k-1\geq 1$. □

LEMMA 3.4
For any configuration x of CA-$58_{0\text{-}1}(n)$, $\delta^k(x)$ *does not contain the subsequence* 000 *where* $k\geq n-2$.

Proof.
We set $\delta^k(x)=y$, $y_iy_{i+1}y_{i+2}=000$ $(1\leq i\leq n-2)$ and $\delta^{k-1}(x)=z$. Then, we have $z_{i+1}z_{i+2}z_{i+3}=000$, 010, 110 or 111 for $y_{i+2}=0$.

(i) In the case $z_{i+1}z_{i+2}z_{i+3}=010$
This case contradicts $y_{i+1}=0$ as $f(*01)=1$.

(ii) In the case $z_{i+1}z_{i+2}z_{i+3}=110$ or 111
For $y_{i+1}=0$, we have $z_i=1$. But this contradicts lemma 3.1 as $k-1\geq n-3$.

(iii) In the case $z_{i+1}z_{i+2}z_{i+3}=000$
Only this case is suitable by $z_{i-1}z_i=00$. Then, we have $z_{i-1}z_iz_{i+1}z_{i+2}z_{i+3}=00000$.

(iii-a) In the case $i-1>n-(i+2)$ i.e. $2i>n-1$
We set $\delta^{k-\{n-(i+2)\}}(x)=w$. Then, we have $w_{n-2}w_{n-1}w_n=000$. As $k-\{n-(i+2)\}>k-(i-1)\geq n-2-(n-3)=1$, we have $k-\{n-(i+2)\}\geq 2$. Therefore, this case contradicts lemma 3.3.

(iii-b) In the case $i-1\leq n-(i+2)$ i.e. $2i\leq n-1$
Considering $x_1x_2x_3=000$ for $y_1y_2y_3=000$ where $\delta(x)=y$, we have $\delta^{k-\{n-(i+2)\}}(x)=0^n$ and $k-\{n-(i+2)\}\geq m-2-(m-3)=1$. But no predecessor of 0^n exists. Therefore, this case is irrelevant. □

LEMMA 3.5
For any configuration x of CA-$58_{0\text{-}1}(n)$, $\delta^k(x)$ *does not contain the subsequence* 111 *where* $k\geq 2n-5$.

Proof.
We set $\delta^k(x)=y$, $y_iy_{i+1}y_{i+2}=111$ $(1\leq i\leq n-2)$, $\delta^{k-1}(x)=z$ and $\delta^{k-2}(x)=w$. Then, we have $z_{i-1}z_iz_{i+1}=001$, 011, 100 or 101 for $y_i=1$.

(i) In the case $z_{i-1}z_iz_{i+1}=001$ or 101
For $y_{i+1}=0$, we have $z_{i+2}=1$. But this contradicts $y_{i+2}=1$ as $f(11*)=0$.

(ii) In the case $z_{i-1}z_iz_{i+1}=011$
This case contradicts $y_{i+1}=1$ as $f(11*)=0$.

(iii) In the case $z_{i-1}z_iz_{i+1}=100$
For $y_{i+1}y_{i+2}=11$, we have $z_{i+2}z_{i+3}=11$ i.e. $z_{i-1}z_iz_{i+1}z_{i+2}z_{i+3}=10011$. Then, we have $w_{i-2}w_{i-1}w_i=001$, 011, 100 or 101 for $z_{i-1}=1$.

(iii-a) In the case $w_{i-2}w_{i-1}w_i=001$ or 101
For $z_i=0$, we have $w_{i+1}=0$. But this contradicts $z_{i+1}=0$ as $f(10*)=1$.

(iii-b) In the case $w_{i-2}w_{i-1}w_i=100$
For $z_i=0$, we have $w_{i+1}=0$. But this contradicts lemma 3.4 as $k-2\geq 2n-7$.

(iii-c) In the case $w_{i-2}w_{i-1}w_i=011$
For $z_{i+1}z_{i+2}=01$, we have $w_{i+1}w_{i+2}=10$ considering $f(10*)=0$. For $z_{i+3}=1$, we have $w_{i+3}w_{i+4}=01$ or 11. Therefore, we have $w_{i-2}w_{i-1}\cdots w_{i+4}=0111001$ or 0111011. Repeating this operation, we set

$\delta^{k-2(i-1)}(x)=u$. Then, we have $u_1u_2u_3=111$. But this contradicts corollary 3.2 because of $k-2(i-1) \geq 2n-5-2i+2 \geq 1$. □

LEMMA 3.6

For any configuration x of CA-$58_{0\text{-}1}(n)$, $\delta^k(x)$ *does not contain the subsequence* 00 *where* $k \geq 2n-4$.

Proof.

We set $\delta^k(x)=y$, $y_iy_{i+1}=00$ $(1 \leq i \leq n-1)$ and $\delta^{k-1}(x)=z$. Considering lemma 3.4 and lemma 3.5, we have $z_{i-1}z_iz_{i+1}=010$ or 110 for $y_i=0$ as $k-1 \geq 2n-5$. But this contradicts $y_{i+1}=0$ as $f(10*)=1$. □

LEMMA 3.7

For any configuration x of CA-$58_{0\text{-}1}(n)$, $\delta^k(x)$ *does not contain the subsequence* 1010 *where* $k \geq 2n-6$.

Proof.

We set $\delta^k(x)=y$, $y_iy_{i+1}y_{i+2}y_{i+3}=1010$ $(1 \leq i \leq n-3)$ and $\delta^{k-1}(x)=z$. For $y_i=1$, we have $z_{i-1}z_iz_{i+1}=001$, 011, 100 or 101.

(i) In the case $z_{i-1}z_iz_{i+1}=001$ or 101
For $y_{i+1}=1$, we have $z_{i+2}=0$. For $y_{i+2}\,y_{i+3}=10$, we have $z_{i+3}z_{i+4}=10$ considering lemma 3.4. Thus, we have $z_{i+1}z_{i+2}z_{i+3}z_{i+4}=1010$.

(ii) In the case $z_{i-1}z_iz_{i+1}=100$
For $y_{i+1}=0$, we have $z_{i+2}=0$. But this contradicts lemma 3.4 as $k-1 \geq 2n-7$.

(iii) In the case $z_{i-1}z_iz_{i+1}=011$
For $y_{i+1}y_{i+2}=01$, we have $z_{i+2}z_{i+3}=01$ considering $f(11*)=0$ and lemma 3.4. For $y_{i+3}=0$, we have $z_{i+4}=0$. Therefore we have $z_{i+1}z_{i+2}z_{i+3}z_{i+4}=1010$. Repeating the above, we set $\delta^{k-\{n-(i+3)\}}(x)=w$. Then, we have $w_{n-3}w_{n-2}w_{n-1}w_n=1010$ where $k-\{n-(i+3)\} \geq 2n-6-(n-4) \geq n-2$. Moreover we set $\delta^{k-\{n-(i+3)\}-1}(x)=u$. For $w_{n-1}w_n=10$, we have $u_{n-2}u_{n-1}u_n=011$. But this contradicts $w_{n-2}=0$ as $f(*01)=1$. □

COROLLARY 3.8

For any configuration x of CA-$58_{0\text{-}1}(n)$, $\delta^k(x)$ *does not contain the subsequence* 0101 *where* $k \geq 2n-5$.

Proof.

We set $\delta^k(x)=y$, $y_iy_{i+1}y_{i+2}y_{i+3}=0101$ $(1 \leq i \leq n-3)$ and $\delta^{k-1}(x)=z$.

(i) In the case $i \geq 2$
We have $z_iz_{i+1}z_{i+2}z_{i+3}=1010$ considering lemma 3.4 and $f(11*)=0$. But this contradicts lemma 3.7 as $k-1 \geq 2n-6$.

(ii) In the case $i=1$
We set $z_1z_2z_3z_4=0010$ and $\delta^{k-2}(x)=w$. Then, we have $w_1w_2w_3w_4\;w_5=00010$. But this contradicts lemma 3.4 as $k-2 \geq 2n-7$. □

LEMMA 3.9

For any configuration x of CA-$58_{0\text{-}1}(n)$, *we set* $\delta^k(x)=y$ *where* $k \geq 2n-4$. *Then, the followings hold*;

1. $y=[110]^*_n$, $[101]^*_n$ *or* $[011]^*_n$.
2. *Above three configurations make a limit cycle*.

Proof.

Considering lemma 3.4, lemma 3.5, lemma 3.6 lemma 3.7 and corollary 3.8, we can construct the configuration *y* as follows ;

・ 011011011・・・ = $[011]^*_n$
・ 101101101・・・ = $[101]^*_n$
・ 110110110・・・ = $[110]^*_n$

It follows that
$\delta([011]^*_n) = [110]^*_n$, $\delta([110]^*_n) = [101]^*_n$ and $\delta([101]^*_n) = [011]^*_n$ by direct calculations.

Especially, we set $x=[1110]^*_n$. Then, we can show that $\delta^{2i}(x)=(110]^*_i\,[1110)^*_{n-i}$ by induction on *i*. Let $i=n-3$, then we have $\delta^{2(n-3)}(x)=(110]^*_{n-2}111$, $\delta^{2n-5}(x)=(101]^*_{n-3}100$ and $\delta^{2n-4}(x)=(011]^*_{n-3}011=(011]^*_n$

$$=\begin{cases} (011)^l=[011]^*_n & n=3l \\ 1(011)^l=[101]^*_n & n=3l+1 \\ 11(011)^l=[110]^*_n & n=3l+2. \end{cases}$$

And the last configurations are configurations on a limit cycle by lemma 3.9. □

Finally, we have the following theorem about *CA*-$58_{0\text{-}1}(n)$.

THEOREM 3.10

CA-$58_{0\text{-}1}(n)$ *has a unique limit cycle of period length* 3 *and its transient length is* $2n-4$.

4. CONCLUSIONS

As conclusive remarks, we could show that

CA-$58_{0\text{-}1}(n)$ has a unique limit cycle of period length 3 and its transient length is $2n$-4. There are some future tasks. The first is to investigate CA-58 (n) under the other fixed boundary conditions 0-0, 1-0 and 1-1. The second is to investigate 1-dimensional cellular automata with threshold rules under free boundary conditions[14]. And the last is to investigate 1-dimensional cellular automata with reversible transition functions considered as a special type of quantum cellular automata[15] and 2-dimensional cellular automata with reversible linear transition functions using von Neumann neighborhood.

REFERENCES

1) J. von Neumann: Theory of Self-Reproducing Automata, Illinois Univ. Press, edited and completed by A. W. Burks, 1966.
2) Stephen Wolfram.: Universality and complexity in cellular automata., *Physica D*, Vol.10, pp.1-35,1984.
3) Stephen Wolfram: Theory and Applications of Cellular Automata, *World Scientific*, 1986.
4) Inokuchi, S. and Kawahara, Y.: On cellular automata with limit cycles of period length one or two (in Japanese), *RIMS Kokyuroku*, Research Institute for Mathematical Sciences Kyoto Univ.,Vol.906,pp.126-134,1995.
5) Sato, T.: On Behaviors of Cellular Automata with Rule 27, Kyushu Journal of Mathematics,Vol.50,No.1,pp.133-152, 1996.
6) Sato, T.: On Behaviors of Cellular Automata with Rule 11, Research Reports of Oita National College of Technology, Vol.32,pp.128-136,1996.
7) Sato, T.: On Behaviors of Cellular Automata with Rule 43(in Japanese), Research Reports of Oita National College of Technology, Vol.33,pp.63-68,1997.
8) Sato, T.: On Behaviors of Cellular Automata with Rule 189, Research Reports of Oita National College of Technology, Vol.34,pp.50-53,1998.
9) Sato, T.: On Behaviors of Cellular Automata with Rule 190, Research Reports of Oita National College of Technology, Vol.35,pp.53-58,1999.
10) Sato, T.: On Behaviors CA-$57_{0\text{-}0}(m)$ and CA-$57_{1\text{-}0}(m)$, Memoirs of Oita National College of Technology, Vol.37, pp.47-51,2001.
11) Sato, T.: On Behaviors CA-$57_{0\text{-}1}(m)$ and CA-$57_{1\text{-}0}(m)$, Memoirs of Oita National College of Technology, Vol.41, pp.63-70,2005.
12) Shingai, R.: The Maximum Period Realized in 1-D Uniform Neural Networks with a Refractory Period, *THE TRANSACTIONS OF THE IECE OF JAPAN*, Vol.E-61, No.10,pp.804-808,1978.
13) Inokuchi, S. and Sato, T.: On Limit Cycles and Transient Lengths of Cellular Automata with Threshold Rules, Bulletin of Informatics and Cybernetics,Vol.32,No.1, pp.25-60,2000.
14) Sato,H. and Nishi,T.: One-dimensional Discrete-time Binary Cellular Neural Networks (in Japanese), *THECNICAL REPORT OF IEICE.*,Vol. NLP200-130,pp.1-8,2001.
15) Inokuchi, S., Honda, K., Lee.H.Y., Sato, T., Mizoguchi, Y. and Kawahara, Y.: On Reversible Cellular Automata with Finite Cell Array, *Lecture Notes in Computer Science*, Springer-Verlag.Vol.3699,pp.130-141,2005.

(Received September 29, 2006)